

ON THE SUM RELATION OF MULTIPLE HURWITZ ZETA FUNCTIONS

CHAN-LIANG CHUNG

ABSTRACT. In this paper we shall define a special-valued multiple Hurwitz zeta functions, namely the multiple t -values $t(\alpha)$ and define similarly the multiple star t -values as $t^*(\alpha)$. Then we consider the sum of all such multiple (star) t -values of fixed depth and weight with even argument and prove that such a sum can be evaluated when the evaluations of $t(\{2m\}^n)$ and $t^*(\{2m\}^n)$ are clear. We give the evaluations of them in terms of the classical Euler numbers through their generating functions.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be a k -tuple positive integer, we define the multiple t -values of depth k [3, 4] by

$$t(\alpha) = t(\alpha_1, \alpha_2, \dots, \alpha_k) = \sum_{1 \leq j_1 < j_2 < \dots < j_k} \frac{1}{(2j_1 - 1)^{\alpha_1} (2j_2 - 1)^{\alpha_2} \dots (2j_k - 1)^{\alpha_k}},$$

which is equal to the multiple Hurwitz zeta functions $2^{-|\alpha|} \zeta(\alpha_1, \dots, \alpha_k; -\frac{1}{2}, \dots, -\frac{1}{2})$ having weight $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k$. Let $(\{a\}^n)$ be the string (a, a, \dots, a) for any positive integer a . It is straightforward that

$$1 + \sum_{n=1}^{\infty} t(\{m\}^n) x^{mn} = \prod_{j=1}^{\infty} \left(1 + \frac{x^m}{(2j-1)^m} \right).$$

Similarly, we can define the multiple star t -values of depth k and weight $|\alpha|$ by

$$t^*(\alpha_1, \alpha_2, \dots, \alpha_k) = \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_k} \frac{1}{(2j_1 - 1)^{\alpha_1} (2j_2 - 1)^{\alpha_2} \dots (2j_k - 1)^{\alpha_k}}.$$

The only change consists in considering the non-strict inequalities under the summation sign. The generating function of $t^*(\{m\}^n)$ is given by

$$\prod_{j=1}^{\infty} \left(1 - \frac{x^m}{(2j-1)^m} \right)^{-1}.$$

That is,

$$1 + \sum_{n=1}^{\infty} t^*(\{m\}^n) x^{mn} = \prod_{j=1}^{\infty} \left(1 - \frac{x^m}{(2j-1)^m} \right)^{-1}.$$

In this paper, we consider the sum of all multiple t -value of depth k and weight mn with argument $m \geq 2$ as

$$T(mn, k) = \sum_{|\alpha|=n} t(m\alpha_1, m\alpha_2, \dots, m\alpha_k).$$

Date: Sep. 5, 2016.

Key words and phrases. Hurwitz zeta function, Multiple zeta value, Multiple zeta star value, Sum formula, Generating functions, Infinite series and products.

This is equivalent to

$$2^{mn}T(mn, k) = \sum_{|\alpha|=n} \zeta \left(m\alpha_1, m\alpha_2, \dots, m\alpha_k; -\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2} \right),$$

and we put

$$T^*(mn, k) = \sum_{|\alpha|=n} t^*(m\alpha_1, m\alpha_2, \dots, m\alpha_k).$$

There is a simple connection between the evaluations of $T(mn, k)$ and $T^*(mn, k)$ and it could be done by a combinatorial argument that is essentially the same as the proof of Lemma 1 in [2].

Lemma 1.1. *For positive integers $k \leq n$ and $m \geq 2$, we have*

$$T^*(mn, k) = \sum_{r=1}^k \binom{n-r}{k-r} T(mn, r).$$

Next we prove that the evaluations of $T(mn, k)$ and $T^*(mn, k)$ are based on the evaluations of multiple t -value $t(\{m\}^p)$ and star t -value $t^*(\{m\}^q)$.

Theorem 1.2. *For positive integers $k \leq n$ and $m \geq 2$, we have*

$$T(mn, k) = \sum_{p=k}^n (-1)^{p-k} \binom{p}{k} t(\{m\}^p) t^*(\{m\}^{n-p})$$

and

$$T^*(mn, k) = \sum_{q=k}^n (-1)^{n+q} \binom{q}{k} t(\{m\}^{n-q}) t^*(\{m\}^q).$$

By Theorem 1.2 and the evaluations of $\zeta(\{2\}^n)$ and $\zeta^*(\{2\}^n)$ in terms of the classical Euler numbers given in Section 3 (formulas (3.1) and (3.2)), we have for positive integers $k \leq n$,

$$\begin{aligned} T(2n, k) &= \frac{(-1)^{n-k} \pi^{2n}}{4^n (2n)!} \sum_{p=k}^n \binom{2n}{2p} \binom{p}{k} E_{2n-2p}; \\ T^*(2n, k) &= \frac{(-1)^n \pi^{2n}}{4^n (2n)!} \sum_{q=k}^n \binom{2n}{2q} \binom{q}{k} E_{2q}. \end{aligned} \tag{1.1}$$

On the other hand, by Lemma 1.1 we also have

$$\begin{aligned} T^*(2n, k) &= \sum_{r=1}^k \binom{n-r}{k-r} T(2n, r) \\ &= \frac{(-1)^n \pi^{2n}}{4^n (2n)!} \sum_{r=1}^k (-1)^r \binom{n-r}{k-r} \sum_{p=r}^n \binom{2n}{2p} \binom{p}{r} E_{2n-2p}. \end{aligned}$$

Therefore there is an Euler-numbers identity behind the two evaluations of $T^*(2n, k)$:

$$\sum_{r=1}^k (-1)^r \binom{n-r}{k-r} \sum_{p=r}^n \binom{2n}{2p} \binom{p}{r} E_{2n-2p} = \sum_{q=k}^n \binom{2n}{2q} \binom{q}{k} E_{2q}.$$

Additionally, we list the evaluations of $T(4n, k)$ and $T^*(4n, k)$ as follows

$$T(4n, k) = \frac{(-1)^n \pi^{4n}}{4^n (4n)!} \sum_{p=0}^{n-k} \frac{(-1)^{p+k}}{4^p} \binom{n-p}{k} \binom{4n}{4p} \sum_{\ell_1 + \ell_2 = p} (-1)^{\ell_1} \binom{4p}{2\ell_1} E_{2\ell_1} E_{2\ell_2};$$

$$T^*(4n, k) = \frac{(-1)^n \pi^{4n}}{4^n (4n)!} \sum_{q=k}^n \frac{(-1)^q}{4^q} \binom{q}{k} \binom{4n}{4q} \sum_{\ell_1 + \ell_2 = q} (-1)^{\ell_1} \binom{4q}{2\ell_1} E_{2\ell_1} E_{2\ell_2}.$$

2. PROOF OF THEOREM 1.2

Following [1], for two real variables y and z , we form the infinite product

$$T_m(x; y, z) = \prod_{n=1}^{\infty} \left(1 + \frac{yx^m}{(2n-1)^m} \right) \left(1 - \frac{zx^m}{(2n-1)^m} \right)^{-1}.$$

Notice that the right hand side of above product are the product of two generating functions of $y^n t(\{m\}^n)$ and $z^n t^*(\{m\}^n)$, respectively.

Proof of Theorem 1.2. It is easy to see that

$$T_m(x; y, z) = 1 + \sum_{n=1}^{\infty} \sum_{p+q=n} y^p z^q t(\{m\}^p) t^*(\{m\}^q) x^{mn},$$

here in convention we let $t(\{m\}^0) = t^*(\{m\}^0) = 1$.

On the other hand, we obtain

$$T_m(x; y, z) = \prod_{n=1}^{\infty} \left[1 + (y+z) \frac{x^m}{(2n-1)^m} + z(y+z) \frac{x^{2m}}{(2n-1)^{2m}} + \cdots \right. \\ \left. + z^{k-1}(y+z) \frac{x^{km}}{(2n-1)^{km}} + \cdots \right],$$

or

$$T_m(x; y, z) = 1 + \sum_{n=1}^{\infty} \sum_{r=1}^n z^{n-r} (y+z)^r T(mn, r) x^{mn}.$$

This implies immediately that

$$(2.1) \quad \sum_{r=1}^n z^{n-r} (y+z)^r T(mn, r) = \sum_{p+q=n} y^p z^q t(\{m\}^p) t^*(\{m\}^q).$$

Applying the differential operator $(\partial^k / \partial y^k)$ to the both sides of above equation and then setting $y = -1, z = 1$ to get

$$k! T(mn, k) = \sum_{p+q=n} \frac{p!}{(p-k)!} (-1)^{p-k} t(\{m\}^p) t^*(\{m\}^q).$$

Hence our first assertion of Theorem 1.2 follows.

If we apply the differential operator $(\partial^k / \partial z^k)$ to the both sides of equation (2.1) and take $y = 1, z = -1$ afterwards, then

$$\sum_{r=1}^k \binom{n-r}{k-r} (-1)^{n-k} T(mn, r) = \sum_{p+q=n} (-1)^{q-k} \binom{q}{k} t(\{m\}^p) t^*(\{m\}^q).$$

By Lemma 1.1, we obtain the second assertion. □

Taking values $y = 0$ and $z = 1$ into equation (2.1) gives the following result.

Corollary 2.1. *For a pair of positive integers n, m with $m \geq 2$, we have*

$$t^*(\{m\}^n) = T(mn, 1) + T(mn, 2) + \cdots + T(mn, n).$$

3. EVALUATIONS OF $t(\{2m\}^n)$ AND $t^*(\{2m\}^n)$

Theorem 1.2 says that the formula of $T(2mn, k)$ can be deduced directly from the evaluations of $t(\{2m\}^n)$ and $t^*(\{2m\}^n)$. J. Zhao [4] gave the evaluation of $t(\{2\}^n)$ for any positive integer n as follows

$$(3.1) \quad t(\{2\}^n) = \frac{\pi^{2n}}{4^n(2n)!}.$$

Then he used the theory of symmetric functions established by M. Hoffman [2] to calculate that

$$(3.2) \quad t^*(\{2\}^n) = \frac{(-1)^n E_{2n} \pi^{2n}}{4^n(2n)!},$$

and for positive integers $k \leq n$,

$$T(2n, k) = \frac{(-1)^{n-k} \pi^{2n}}{4^n(2n)!} \sum_{\ell=0}^{n-k} \binom{n-\ell}{k} \binom{2n}{2\ell} E_{2\ell},$$

where E_{2n} is the $2n$ -th Euler number defined by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \quad \text{for } |x| < \frac{\pi}{2}.$$

This is equivalent to the formula given by (1.1) in Section 1.

According to the parity of m , we divide the general evaluations of $t(\{2m\}^n)$ by two cases.

Proposition 3.1. *For positive integers n and m with $m \geq 3$ odd, we let $w_m = e^{\frac{2\pi i}{m}}$ and have*

$$t(\{2m\}^n) = \frac{\pi^{2mn}}{2^{m-1}(2mn)!} \sum_{k=1}^{(m-1)/2} \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq m-1} (w_m^{j_1} + w_m^{j_2} + \cdots + w_m^{j_k})^{2mn}.$$

Proof. For $x \neq 0$ and $|x| < 1$, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} (-1)^n t(\{2m\}^n) x^{2mn} &= \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{(2j-1)^{2m}} \right) \\ &= \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{j^{2m}} \right) / \left(1 - \frac{x^{2m}}{(2j)^{2m}} \right). \end{aligned}$$

Note that

$$\begin{aligned} \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{j^{2m}} \right) &= \prod_{j=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \frac{(w_m^k x)^2}{j^2} \right) \\ &= \prod_{k=0}^{m-1} \frac{\sin(w_m^k \pi x)}{w_m^k \pi x} \\ &= \frac{1}{(\pi x)^m} \prod_{k=0}^{m-1} \sin(w_m^k \pi x). \end{aligned}$$

Let $y = \pi x/2$. Thus,

$$\prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{j^{2m}}\right) / \left(1 - \frac{x^{2m}}{(2j)^{2m}}\right) = \prod_{k=0}^{m-1} \cos(w_m^k y).$$

We express the product of cosine functions into a linear combination of cosine functions as

$$\frac{1}{2^{m-1}} \sum_{\varepsilon_j = \pm 1, 1 \leq j \leq m-1} \cos(w_m^{m-1} + \varepsilon_1 w_m^{m-2} + \cdots + \varepsilon_{m-1})y.$$

It can be rewritten as

$$\frac{1}{2^{m-1}} \sum_{k=1}^{(m-1)/2} \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq m-1} \cos(2y(w_m^{j_1} + w_m^{j_2} + \cdots + w_m^{j_k})),$$

or

$$\frac{1}{2^{m-1}} \sum_{k=1}^{(m-1)/2} \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq m-1} \cos(\pi x(w_m^{j_1} + w_m^{j_2} + \cdots + w_m^{j_k})),$$

since $w_m^m = 1$ and for $m \geq 3$ that $w_m^{m-1} + w_m^{m-2} + \cdots + w_m + 1 = 0$. Extracting the coefficient of x^{2mn} from the expression leads to the evaluation of $t(\{2m\}^n)$. \square

Proposition 3.2. *For positive integers n and m with $m \geq 2$ even, we let $w = w_{2m} = e^{\frac{2\pi i}{2m}}$ and have*

$$t(\{2m\}^n) = \frac{(-1)^n \pi^{2mn}}{2^{2mn+m-2} (2mn)!} \operatorname{Re} \left(\sum_{\varepsilon \in A} (w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})^{2mn} \right),$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1})$ and for each $1 \leq j \leq m-1$ we have either $\varepsilon_j = 1$ or $\varepsilon_j = -1$. Here A is the set of elements of the form $w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1}$ such that the number of -1 in ε is even.

Proof. As in the proof of Proposition 3.1, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} (-1)^n t(\{2m\}^n) x^{2mn} &= \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{(2j-1)^{2m}}\right) \\ &= \prod_{k=0}^{m-1} \cos(w^k y), \end{aligned}$$

where $y = \pi x/2$. Now we express the product of cosine functions into a linear combination of cosine functions

$$\prod_{k=0}^{m-1} \cos(w^k y) = \frac{1}{2^{m-1}} \sum_{\varepsilon_j = \pm 1, 1 \leq j \leq m-1} \cos(w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})y.$$

It immediately follows that

$$(3.3) \quad t(\{2m\}^n) = \frac{(-1)^n}{2^{m-1}} \sum_{\varepsilon_j = \pm 1, 1 \leq j \leq m-1} \frac{(-1)^{mn} \pi^{2mn}}{(2mn)! 2^{2mn}} (w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})^{2mn}.$$

For any $B_j = w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1} \in A^c$, where A^c denote the complement of the set A . If $\varepsilon_{m-2} = 1$, then

$$\begin{aligned} -\overline{B_j} &= w^{\frac{m}{2}} \left(w^{\frac{m}{2}-1} + \varepsilon_{m-3} w^{\frac{m}{2}-2} + \cdots + \varepsilon_1 w^{2-\frac{m}{2}} + w^{1-\frac{m}{2}} + i\varepsilon_{m-1} \right) \\ &= w^{m-1} + \varepsilon_{m-3} w^{m-2} + \cdots + \varepsilon_1 w^2 + w - \varepsilon_{m-1}. \end{aligned}$$

It implies $-\overline{B_j} \in A$ since the number of -1 in $\varepsilon_{m-3}, \varepsilon_{m-4}, \dots, \varepsilon_1, 1, -\varepsilon_{m-1}$ is even.

If $\varepsilon_{m-2} = -1$, then

$$\begin{aligned}\overline{B_j} &= -w^{\frac{m}{2}} \left(-w^{\frac{m}{2}-1} + \varepsilon_{m-3} w^{\frac{m}{2}-2} + \dots + \varepsilon_1 w^{2-\frac{m}{2}} + w^{1-\frac{m}{2}} + i\varepsilon_{m-1} \right) \\ &= w^{m-1} - \varepsilon_{m-3} w^{m-2} - \dots - \varepsilon_1 w^2 - w + \varepsilon_{m-1}.\end{aligned}$$

Since the number of -1 in $-\varepsilon_{m-3}, -\varepsilon_{m-4}, \dots, -\varepsilon_1, -1, \varepsilon_{m-1}$ is even, we have $\overline{B_j} \in A$. Thus, there is a one-to-one corresponding from A^c to A . Let $A = \{A_1, A_2, \dots, A_{2^{m-2}}\}$ and $A^c = \{B_1, B_2, \dots, B_{2^{m-2}}\}$, then

$$\begin{aligned}& \sum_{\varepsilon_j = \pm 1, 1 \leq j \leq m-1} (w^{m-1} + \varepsilon_1 w^{m-2} + \dots + \varepsilon_{m-1})^{2mn} \\ &= \sum_{j=1}^{2^{m-2}} (A_j^{2mn} + B_j^{2mn}) \\ &= \sum_{j=1}^{2^{m-2}} (A_j^{2mn} + \overline{A_j}^{2mn}) \\ &= 2 \operatorname{Re} \left(\sum_{\varepsilon \in A} (w^{m-1} + \varepsilon_1 w^{m-2} + \dots + \varepsilon_{m-1})^{2mn} \right).\end{aligned}$$

From which and (3.3) our assertion follows. \square

By Proposition 3.1 and 3.2, it immediately follows for any positive integer n that

$$\begin{aligned}t(\{4\}^n) &= \frac{\pi^{4n}}{4^n (4n)!}, \quad t(\{6\}^n) = \frac{3\pi^{6n}}{4 \cdot (6n)!} \quad \text{and} \\ t(\{8\}^n) &= \frac{\pi^{8n}}{2 \cdot (8n)!} \left[\left(1 + \frac{1}{\sqrt{2}}\right)^{4n} + \left(1 - \frac{1}{\sqrt{2}}\right)^{4n} \right].\end{aligned}$$

There is also a slight difference when m is even or odd in the general formula of $t^*(\{2m\}^n)$.

Proposition 3.3. *For positive integers n, m with m odd, we have*

$$t^*(\{2m\}^n) = \frac{(-1)^n \pi^{2mn}}{2^{2mn}} \sum_{|\ell|=mn} \prod_{j=0}^{m-1} \frac{E_{2\ell_j}}{(2\ell_j)!} w_m^{2j\ell_{j+1}},$$

where $w_m = e^{\frac{2\pi i}{m}}$ and the summation ranges over all nonnegative integers $\ell_1, \ell_2, \dots, \ell_m$ such that $\ell_1 + \ell_2 + \dots + \ell_m = mn$.

Proof. It is straightforward that

$$\begin{aligned}1 + \sum_{n=1}^{\infty} t^*(\{2m\}^n) x^{2mn} &= \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{(2j-1)^{2m}} \right)^{-1} \\ &= \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{j^{2m}} \right)^{-1} \bigg/ \left(1 - \frac{x^{2m}}{(2j)^{2m}} \right)^{-1}.\end{aligned}$$

Let $w_m = e^{\frac{2\pi i}{m}}$. Note that

$$\begin{aligned} \prod_{j=1}^{\infty} \left(1 - \frac{x^{2m}}{j^{2m}}\right)^{-1} &= \prod_{j=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \frac{(w_m^k x)^2}{j^2}\right)^{-1} \\ &= w_m^{\frac{m(m-1)}{2}} (\pi x)^m \prod_{k=0}^{m-1} \csc(w_m^k \pi x). \end{aligned}$$

Thus we have

$$1 + \sum_{n=1}^{\infty} t^*(\{2m\}^n) x^{2mn} = \prod_{k=0}^{m-1} \sec\left(\frac{w_m^k \pi x}{2}\right).$$

Comparing the coefficient of x^{2mn} of the above equation gives the desired evaluation of $t^*(\{2m\}^n)$ for m is odd. \square

Proposition 3.3 implies that, in particular when $m = 1$, the formula (3.2). In addition, we have

$$t^*(\{6\}^n) = \frac{(-1)^n \pi^{6n}}{2^{6n}} \sum_{|\ell|=3n} \frac{E_{2\ell_1} E_{2\ell_2} E_{2\ell_3}}{(2\ell_1)!(2\ell_2)!(2\ell_3)!} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{2\ell_2+4\ell_3}.$$

Proposition 3.4. *Let $w = e^{\frac{2\pi i}{m}}$. For positive integers n, m with m even, we have*

$$t^*(\{2m\}^n) = \frac{\pi^{2mn}}{2^{2mn}} \sum_{|\ell|=mn} \prod_{j=0}^{m-1} \frac{E_{2\ell_j}}{(2\ell_j)!} w^{2j\ell_{j+1}}.$$

Proof. As in the proof of Theorem 3.3, we have

$$1 + \sum_{n=1}^{\infty} t^*(\{2m\}^n) x^{2mn} = \prod_{k=0}^{m-1} \sec\left(\frac{w^k \pi x}{2}\right),$$

where $w = e^{\frac{2\pi i}{2m}}$ in this case. From which we extract the coefficient of x^{2mn} to obtain the desired evaluation of $t^*(\{2m\}^n)$ for even m . \square

For example, we have

$$\begin{aligned} t^*(\{4\}^n) &= \frac{\pi^{4n}}{2^{4n}(4n)!} \sum_{\ell=0}^{2n} (-1)^\ell \binom{4n}{2\ell} E_{2\ell} E_{4n-2\ell}; \\ t^*(\{8\}^n) &= \frac{\pi^{8n}}{2^{8n}} \sum_{|\ell|=4n} \frac{E_{2\ell_1} E_{2\ell_2} E_{2\ell_3} E_{2\ell_4}}{(2\ell_1)!(2\ell_2)!(2\ell_3)!(2\ell_4)!} i^{\ell_2+2\ell_3+3\ell_4}. \end{aligned}$$

REFERENCES

- [1] K.-W. Chen, C.-L. Chung and M. Eie, *Sum formulas of multiple zeta values with arguments are multiples of a positive integer*, submitted, arXiv:1608.01412.
- [2] M. E. Hoffman, *On multiple zeta values of even arguments*, Int. J. Number Theory, to appear; arXiv:1205.7051v4 (2016).
- [3] Z. Shen, T. Cai, *Some identities for multiple Hurwitz zeta values*, (in Chinese) Sci Sinica Math. **41** (2011), 955–970.
- [4] J. Zhao, *Sum formula of multiple Hurwitz-zeta values*, Forum Math. **27** (2015), 929–936.

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, 6F, ASTRONOMY-MATHEMATICS BUILDING, NO. 1, SEC. 4, ROOSEVELT ROAD, TAIPEI 10617, TAIWAN(R.O.C.)
E-mail address: andrechung@gate.sinica.edu.tw